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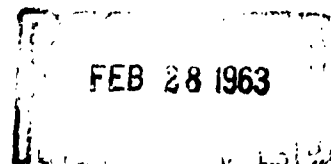# CONTINUOUSLY EQUIVALENT NETWORKS
AND THEIR APPLICATION

By

James D. Schoeffler

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CONTINUOUSLY EQUIVALENT NETWORKS AND THEIR APPLICATION

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James D. Schoeffler

Engineering Division

Case Institute of Technology

Cleveland 6, Ohio

ABSTRACT

The design of filters and amplifiers with minimum element distribu-
tion, minimum number of elements, minimum sensitivity to parameter change,
etc. is important today. Since the synthesis problem usually has an
infinite number of solutions, it is desirable to be able to transform a
given design into a network with the same transfer function but having
fewer elements or better element distribution, etc. The techniques of
equivalent network theory which apply a congruence transformation to a
network matrix in order to generate equivalent networks are used to do
this. An extension of the Cauer formulation is discussed which applies
the transformation directly to the element parameter matrices, thereby
placing the elements of the transformed network directly in evidence.
The transformation is then used to transform to an equivalent network
whose elements differ from those in the original network by only an in-
cremental amount. In the limiting case, a set of differential equations
for the elements of the equivalent network result with arbitrary inputs.
The method of steepest descent is applied to choose these inputs in order
to force the network to converge on the optimal design.

CONTINUOUSLY EQUIVALENT NETWORKS AND THEIR APPLICATION

By

J.D. Schoeffler[†]

I. Introduction

Among the major goals of network theory are the elimination of
transformers from networks and the control of element-value distribution,
topology, and sensitivity of the network to changes in element values.[1]
Usually these goals are approached by demanding that a synthesis technique
be found which yields a network with the desirable qualities listed above.
Hence a great number of different synthesis techniques have been found, no
one of which is general enough to attain all of the listed goals. For
example, the Brune and Darlington synthesis methods yield networks with a
minimum number of elements but have mutual coupling among the elements.
The Bott-Duffin method requires no transformers but uses an extremely large
number of elements.

A different design philosophy was propounded by Cauer in 1929[2] and
Howitt[3] in 1931. They found that a whole family of networks could be
generated from a given network by means of matrix transformations. If
these transformations were properly selected, all of the networks so found
were equivalent, that is, had the same driving point and/or transfer
impedance as the original network. This result implies that the great
burden of achieving all of the listed desirable goals can be removed from
the synthesis technique. The synthesis method need merely find a single
suitable network. Then, from the group of equivalent networks found by
matrix transformation, a network is selected which has no mutual coupling,
and does have good element distribution, topology, etc.

Unfortunately, the matrix transformation method never realized its potentiality despite much effort.[4,5,6] The problem is that there is no unique relation between network topology, elements, and parameter matrices. Hence, as soon as the original parameter matrix is transformed, it becomes very difficult to recognize the new network. In particular, it is difficult to see in advance how to choose the transformation matrix so that desirable elements result. Only the simplest applications of this method have thus far been found.

In this paper, an extension of the Cauer formulation of the equivalent network problem is discussed which applies the transformations directly to the element parameter matrices, thereby placing the element of the transformed network directly in evidence. The transformation is then used to transform to an equivalent network whose elements differ from those in the original network by only an incremental amount. In the limit as this incremental amount approaches zero, a set of first order linear differential equations for the elements of the equivalent network result. The elements of the original network then serve as the initial conditions for these equations. The differential equations have some arbitrary input functions such that for any choice of these inputs, the solutions to the equations yield elements which are functions of the independent variable and which for any value of the independent variable constitute an equivalent network. Thus it is possible to transform from a given network to an equivalent network by varying the elements of the original network in a continuous manner. The advantage of this approach is that the elements of the transformation matrices become very easy to choose in contrast to the situation where the transformation is not continuous.

The problem of selecting these arbitrary inputs so that optimal networks result is considered. In particular, the method of steepest descent is applied to design networks with minimum element distribution and/or minimum number of elements.

II. Equivalent Network Theory

Recently it has been shown [7,8,9,10] that the Cauer formulation of the equivalent theory problem[3] could be extended in such a way that direct control of the elements of the equivalent network resulted. To be more explicit, the usual Cauer formulation generates equivalent networks by applying congruence transformations to the terminal admittance matrix of a network. This technique was extended so that transformations can be applied directly to the branch admittance matrix while at the same time preserving a driving point or transfer ratio or some combination of driving point and transfer ratios. These results are used in this paper and are summarized here.

Let the equilibrium equations of a network with n independent mode pairs on the admittance basis be given by

$$I = Y_4 E \qquad\qquad 2.1$$

where I is the nx1 column matrix of current variables and E is the nx1 column matrix of voltage variables and Y_4 is the nxn terminal admittance matrix. To transform to an equivalent network, Cauer applies the transformation:

$$E = A_4 E'$$

$$2.2$$

$$I' = A_4^t I$$

where A_4 is nonsingular and superscript t means transpose. This results in the new set of equilibrium equations

$$I' = Y_4' E'$$ 2.3

where the new terminal admittance matrix is related to the given matrix by a congruence transformation:

$$Y_4' = A_4^t Y_4 A_4$$ 2.4

Equivalence is assured if A_4 is nonsingular and certain elements of A_4 are properly chosen.[3] That is, the new and old networks are assured of being realizable and of having any combination of driving point and/or transfer impedances the same.

By using the generalized equilibrium equations of Guillemin, it was shown that a related transformation could be applied directly to the bxb branch admittance matrix Y_b where Y_b is defined by

$$J = Y_b V$$ 2.5

and J and V are the (bx1) column matrices of branch voltages and currents. Making the transformations

$$V = T V'$$

2.6

$$J' = T^t J$$

yields a new branch admittance matrix Y_b' given by

$$Y_b' = T^t Y_b T$$ 2.7

Since the elements of Y_b and Y_b' are the elements of the original and transformed networks, it is clear that this approach to equivalence gives the elements of the new network explicitly, thereby allowing control over them. Note also that if the original network is passive, the branch admittance matrix is positive definite or semi-definite and that the congruence transformation preserves this characteristic. If the original network has no mutual coupling, Y_b is a diagonal matrix with all positive entries. If T is chosen so that Y_b' is also diagonal, the congruence transformation insures that all of the elements of Y_b' are also positive. Thus no negative elements can appear due to the application of the transformation. This is the problem in the usual Cauer formulation as discussed by Howitt.[3]

The most general branch is a parallel connection of R, L, and C and hence the branch admittance matrix of the network may be written as

$$Y_b = G + sC + \frac{1}{s} L^{-1} \qquad 2.8$$

where G is the branch conductance matrix, C is the branch capacitance matrix, and L^{-1} is the branch inverse inductance matrix. It has been shown that equivalent networks result if congruence transformations are applied to each of the element matrices as follows:

$$G' = T_1^t \, G \, T_1$$

$$C' = T_2^t \, C \, T_2$$

$$(L^{-1})' = T_3^t \, (L^{-1}) \, T_3 \qquad 2.9$$

and if each of the transformations satisfies the following constraint equation

$$T_i \, \alpha^t = \alpha^t \, A_4 \qquad 2.10$$

where α is the usual nxb cut set matrix of the network and A_4 is a non-singular nxn matrix with certain rows properly chosen to keep desired transfer ratios invariant. Notice that different transformations may be applied to each of the element matrices but all must satisfy Eq. 2.10 with the same nonsingular A_4.

If the original network was realizable, the transformed network is also realizable since the congruence transformation leaves positive definite and semidefinite matrices at least positive semidefinite. Inspection of Eq. 2.7 shows that the new network defined by Y_b' has no mutual coupling provided that TtY_bT is diagonal or that, in other words, each of the T_i is orthogonal with respect to the corresponding parameter matrix. The placing in evidence of the elements of the new and old networks in this approach to equivalence is very advantageous. The problem of choosing the arbitrary elements of the transformation in order to arrive at optimal networks is the subject of this paper.

III. Continuously Equivalent Networks

The transformation from one network to another equivalent network can be visualized by considering the vector space interpretation of a network. Let a network have b elements. Then the branch admittance matrix Y_b is square with b^2 entries. For a reciprocal network, Y_b is symmetrical and hence there are only $(b^2 + b)/2$ independent quantities in the matrix. Call this number Q. In a Q-dimensional vector space, each coordinate of which is the value of one entry of Y_b, each point represents a network. The set of all networks equivalent to a given network corresponds to a set of points on a surface in this Q-dimensional space. If an equivalent network has no mutual coupling, only b of the Q entries are nonzero and the set of all equivalent networks without mutual coupling

corresponds to a set of points on a subsurface of the surface of equivalent networks. We are interested in transforming from the original network (a point in Q-space) to a more desirable network, that is, to some other point in Q-space which lies on the surface of equivalent networks.

For simple networks, it is possible to choose transformation matrices which satisfy the equivalence restrictions and which result in networks which are more desirable in some sense. But for a network of even moderate complexity, it is not easy to choose the free parameter of T. With the vector space interpretation in mind, we can imagine the transformation not as a single "jump" from one point on the surface to another, but rather as a succession of transformations T_1, T_2, ... each of which transforms the original point to another point on the surface a differential distance away. In the limit, the transformation T becomes continuously varying and the point on the surface of equivalent networks moves, following a continuous line from the original network to the final network.

To make this approach more concrete, consider an independent variable x. Then if the transformation T is considered as a function of x, $T(x)$ the kth element of the equivalent network e_k is also a function of x, $e_k(x)$. That is, for any value of x, the set $e_1(x)$, $e_2(x)$, ...,$e_b(x)$ is equivalent to the original network. Using this approach, the selection of the transformation matrix T is greatly facilitated.

To formulate the problem on a continuous basis, consider the transformation matrix T which transforms Y_b into $Y_b + \Delta Y_b$ where ΔY_b is incremental in size:

$$T = U + \Delta T \qquad\qquad 3.1$$

where U is the bxb unit matrix. The application of T to Y_b results in Y_b' given by

$$Y_b' = T^t Y_b T$$

$$= Y_b + Y_b \Delta T + \Delta T^t Y_b + \Delta T^t Y_b \Delta T \qquad 3.2$$

Define

$$T = T_1 \Delta x \qquad 3.3$$

where Δx is a scalar. Then T and Y_b' become

$$T = U + \Delta T = U + T_1 \Delta x$$

$$Y_b' = Y_b + (Y_b T_1 + T_1^t Y_b) \Delta x + 0(\Delta x^2) \qquad 3.4$$

Writing $T = T(x)$, $Y_b = Y_b(x)$, and $Y_b' = Y_b(x + \Delta x)$, Eq. 3.4 becomes

$$\frac{Y_b(x + \Delta x) - Y_b(x)}{\Delta x} = Y_b T_1 + T_1^t Y_b + \frac{0(\Delta x^2)}{\Delta x} \qquad 3.5$$

Passing to the limit yields the matrix differential equation

$$\frac{dY_b}{dx} = Y_b T_1 + T_1^t Y_b \qquad 3.6$$

Let $Y_b = [e_{ij}]$.

Since Y_b is symmetrical Eq. 3.6 is symmetrical and hence correspond to only $(b^2 + b)/2$ equations each of which is of the form

$$\frac{de_{ij}}{dx} = u_i^t [Y_b T_1 + T_1^t Y_b] u_j \qquad \begin{matrix} i = 1, 2, \ldots, b \\ j = i, i+1, \ldots, b \end{matrix} \qquad 3.7$$

where u_k is the kth unit vector in b space.

The matrix T_1 is a function of x, $T_1(x)$, and must be selected so that equivalent networks result. To derive the equivalence restrictions on T_1, start with the general restriction

$$Ta^t = a^t A_4 \qquad 3.8$$

Writing

$$T = U + T_1 \Delta x \qquad 3.9$$

and

$$A_4 = U + B\Delta x \qquad 3.10$$

we have

$$a^t + T_1 a^t \Delta x = a^t + a^t B \Delta x \qquad 3.11$$

or

$$T_1 a^t = a^t B \qquad 3.12$$

Thus the restrictions on T_1 are very similar to those on T except that B is of the form (considering the preservation of the driving point and transfer impedances at terminal pairs 1 and 2):

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \qquad 3.13$$

where the x's indicate arbitrary numbers. That is, B has rows of zeros corresponding to the terminal pairs in question. Writing out Eq. 3.12 yields nb linear equations relating the b^2 entries of T_1 and the $b^2 - kb$ entries of B (k is the number of terminal pairs at which equivalence is retained). Thus fewer than b^2 of the elements of T are independent.

The technique is best illustrated by an example.

Consider the network shown in Fig. 1. We can transform this network by considering it to be two parallel networks each with the graph and tree shown in Fig. 2. For this graph, tree, and branch numbering, the standard cut set matrix is

$$\alpha = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad 3.14$$

If the branch admittance matrices are

$$G = \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \qquad 3.15$$

and

$$L^{-1} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{bmatrix} \qquad 3.16$$

then

$$Y_b = G + \frac{1}{s} L^{-1} \qquad 3.17$$

According to Section 2, these may be transformed to find equivalent networks by congruence transformations T_G and T_L

$$G' = T_G^t \, G \, T_g$$

$$3.18$$

$$(L^{-1})' = T_L^t \, L^{-1} T_L$$

provided that

$$T_G \alpha^t = \alpha^t A_4$$

$$3.19$$

$$T_L \alpha^t = \alpha^t A_4$$

where A_4 is an nxn nonsingular matrix. Assuming only the input impedance is to be preserved, A_4 must have the form

$$A_4 = \begin{bmatrix} 1 & 0 \\ a_1 & a_2 \end{bmatrix} \qquad 3.20$$

with a_2 nonzero.

In the continuous case we have

$$\frac{dG}{dx} = GT_1 + T_1^t G$$

and

$$\frac{dL^{-1}}{dx} = L^{-1} T_2 + T_2^t L^{-1} \qquad 3.21$$

where

$$T_1 \alpha^t = \alpha^t B$$

$$T_2 \alpha^t = \alpha^t B$$

and

$$B = A_4 - U = \begin{bmatrix} 0 & 0 \\ b_1 & b_2 \end{bmatrix} \qquad 3.22$$

In order for A_4 to be nonsingular, b_2 must not equal -1.

Let

$$T_1 = \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \\ t_7 & t_8 & t_9 \end{bmatrix} \qquad 3.23$$

and

$$T_2 = \begin{bmatrix} t_1' & t_2' & t_3' \\ t_4' & t_5' & t_6' \\ t_y' & t_8' & t_9' \end{bmatrix} \qquad 3.24$$

Then Eq. 3.22 give constraint relations among the t_{ij} and among the t_{ij}'. Specifically, each matrix equation yields 6 homogeneous equations in 11 parameters. Hence only five of each set are independent. Solving for the remaining six in terms of these five yields the constraints:

$$
\begin{aligned}
t_2 &= -t_1 - b_1 & \qquad t_2' &= -t_1' - b_1 \\
t_3 &= t_1 - b_2 & \qquad t_3' &= t_1' - b_2 \\
t_5 &= \quad\; - t_4 & \qquad t_5' &= \quad\; - t_4' \\
t_6 &= t_4 & \qquad t_6' &= t_4' \\
t_8 &= -t_7 + b_1 & \qquad t_8' &= -t_7' + b_1 \\
t_9 &= t_7 + b_2 & \qquad t_9' &= t_7' + b_2 & \qquad 3.25
\end{aligned}
$$

The differential equations for the G matrix are subject to a further constraint. That is, all of diagonal elements of G must be zero since it corresponds to a resistor network. The same need not be true for the inductance matrix because mutual coupling is allowed in this case. Hence Eqs. 3.21 becomes

$$\frac{d}{dx} \begin{bmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} = GT_1 + T_1^t G \qquad 3.26$$

or

$$\frac{dg_1}{dx} = 2g_1 t_1$$

$$\frac{dg_2}{dx} = 2g_2 t_5$$

$$\frac{dg_3}{dx} = 2g_3 t_9$$

and

$$0 = g_1 t_1 + g_2 t_9$$

$$0 = g_1 t_3 + g_3 t_7$$

$$0 = g_2 t_6 + g_3 t_8 \qquad 3.27$$

Combining Eq. 3.27 with Eq. 3.25 yields

$$\frac{dg_1}{dx} = 2g_1 t_1$$

$$\frac{dg_2}{dx} = -2g_2 t_4$$

$$\frac{dg_3}{dx} = 2g_3 (t_7 + b_2)$$

$$0 = -g_1 (t_1 + b_1) + g_2 t_4$$

$$0 = g_1 (t_1 - b_2) + g_3 t_7$$

$$0 = g_2 t_4 + g_3 (b_1 - t_7) \qquad 3.28$$

The parameters t_1, t_4, and t_7 may be eliminated from the differential equations using the last three algebraic equations. Solving yields

$$2g_1 t_1 = g_1 (b_2 - b_1) - g_3 b_1$$

$$2g_2 t_4 = g_1 (b_2 + b_1) - g_3 b_1$$

$$2g_3 t_7 = g_1 (b_2 + b_1) + g_3 b_1 \qquad 3.29$$

Hence the differential equations for the resistors become

$$\frac{dg_1}{dx} = g_1(b_2 - b_1) - g_3 b_1$$

$$\frac{dg_2}{dx} = - g_1(b_2 + b_1) + g_3 b_1$$

$$\frac{dg_3}{dx} = g_1(b_2 + b_1) + g_3(b_1 + 2b_2) \qquad 3.30$$

Notice that the algebraic equations in Eqs. 3.28 may not be solvable for b_1 and b_2 in case any element is zero. For example, if $g_1 = 0$ for any value of x, we find

$$0 = g_2 t_4$$

$$0 = g_3 t_7$$

$$0 = g_2 t_4 + g_3(b_1 - t_7) \qquad 3.31$$

or

$$t_4 = t_1 = b_1 = 0$$

Hence in this case, b_1 is no longer independent. Several degenerate cases of interest arise. These cases and the additional constraints they imply are summarized in Table 1. Inspection of Table 1 shows that as an element becomes zero, it must stay zero and the additional constraints can be derived by setting that element equal to zero in Eqs. 3.30. This follows because the equivalent networks are derived from congruence transformations and an element becoming zero causes a decrease in rank of Y_b which cannot then be increased by a congruence transformation.

The differential equations for the elements of the inductance matrix are not so restricted since off-diagonal elements need not be zero. Hence there are 6 differential equations. For the purposes of illustration, suppose that no mutual coupling is allowed. Then only three differential

equations result and they are identical in form to those for the resistors. The resulting set of six differential equations for the network is:

$$g_1' = g_1(b_2 - b_1) - g_3 b_1$$

$$g_2' = - g_1(b_2 + b_1) + g_3 b_1$$

$$g_3' = g_1(b_2 + b_1) + g_3(2b_2 + b_1)$$

$$h_{11}' = h_{21}(b_2 - b_1) - h_{33} b_1$$

$$h_{22}' = - h_{11}(b_2 + b_1) = h_{33} b_1$$

$$h_{33}' = h_{21}(b_2 + b_1) + h_{33}(2b_2 + b_1) \qquad 3.32$$

This set is valid only as long as no element is zero. As an element passes through zero, additional constraints as in Table 1 must be applied. The inputs $b_1(x)$ and $b_2(x)$ are arbitrary functions except that b_2 must never equal -1 so that A_4 remains nonsingular. Continuing the example, suppose the elements of the given network are those in Fig. 3 and suppose we arbitrarily select

$$b_1(x) = 1$$

$$b_2(x) = 0 \qquad 3.33$$

Solution of Eqs. 3.32 with inputs defined by Eqs. 3.33 is easily accomplished by hand. The resulting elements as functions of x are shown in Fig. 4a. The element values are also plotted in Fig. 5 for x between 0 and 1/3. It is easily verified that for any x, the original and transformed networks have the same driving point impedance. At $x = 1/3$, h_{11} becomes zero. Thus inductance number 1 becomes infinite and the constraint $b_1 = 0$ must now be applied. To continue transforming the network, suppose we now choose

$$b_1(x) = 0$$

$$b_2(x) = 1 \qquad\qquad 3.34$$

The resulting elements for $x > 1/3$ are shown in Fig. 4b, and are also plotted in Fig. 5. The inputs selected here were completely arbitrary and merely included for illustrative purposes. The selection of the arbitrary inputs in order to arrive at optimal networks is discussed in the next section.

IV. Optimal Network Design

As mnetioned in the introduction, optimal network design - that is, the design of networks with prescribed driving point and/or transfer ratios plus minimum element distribution and minimum number of elements etc. - is difficult using present synthesis techniques. The following approach is proposed. After a network with a desired transfer function has been designed by network synthesis techniques, the network is transformed continuously to a more optimal network by proper choice of the arbitrary inputs. For example, if the element distribution is to be minimized, the inputs can be selected so that as x increases, the element distribution decreases. Such a procedure requires an explicit error criterion, that is, an explicit measure of the deviation of the given network from the desired optimum. Since solution of the differential equations for the elements is most efficiently handled on a digital comnuter, there is in general no particular advantage to choosing very simple linear criteria. Two examples of error criteria are the following. If element distribution is to be minimized a convenient measure of the deviation from the optimum is a quadratic form:

$$\phi = \sum_{k=1}^{b} (g_k - \bar{g})^2 \qquad\qquad 4.1$$

where $\bar{g}$ is the average of the elements:

$$\bar{g} = \frac{1}{b} \sum_{k=1}^{b} g_k \qquad 4.2$$

Thus $\emptyset$ is a mean square measure of the deviation of the elements from their mean and if $\emptyset$ is minimized, so is the element distribution. Since the g_k are functions of x, $\emptyset$ is also and hence if we calculate $d\emptyset/dx$, it is a function of the elements and their derivatives and consequently a function of the arbitrary inputs. These inputs can then be selected to reduce $\emptyset(x)$ as quickly as possible. This is called the "method of steepest descent" and is an example of a minimization technique that can easily be applied." The similarity to the optimization problems encountered in the field of automatic control should be noted."

Another criterion of interest is the number of elements. For example, in an RL network as in the example, it may be desirable to minimize the number of coupled inductors, etc. A possible measure of the deviation from the optimum for the former case is a weighted sum of the inductance values. Since all the inductances are positive, the weightings can be chosen to emphasize any particular element or group of elements. For the example problem we choose

$$\emptyset(x) = \sum_{k=1}^{3} a_k h_{kk} \qquad 4.3$$

where a_k are arbitrary weighting factors. The derivative of $\emptyset(x)$ with respect to x becomes after substituting the expressions for the derivatives of the elements:

$$\frac{d\emptyset}{dx} = b_1[h_{11}(a_3 - a_1 - a_2) + h_{33}(a_3 - a_1 + a_2)] + b_2[h_{11}(a_3 + a_1 - a_2) + h_{33}(2a_3)]$$

$$4.4$$

Clearly to minimize $d\phi/dx$ (to maximize rate of descent) choose

$$b_1 \left[h_{11}(a_3 - a_1 - a_2) + h_{33}(a_3 - a_1 + a_2) \right] < 0$$

and

$$b_2 \left[h_{11}(a_3 + a_1 - a_2) + h_{33}(2a_3) \right] < 0 \qquad\qquad 4.5$$

Choosing the inputs according to Eq. 4.5 and solving the differential equations is best handled on a machine. In this case, the differential equations are replaced by finite difference approximations, requiring that the inputs be limited in magnitude. According to Eq. 4.4, the arbitrary inputs should always be at their limits with their signs determined from Eq. 4.5. The solution of the set of equations with these inputs is straightforward and rapid on a computer and poses no problems. Notice that the weighting factors are arbitrary and free to be changed during the course of a solution in order to weight certain elements more heavily at the discretion of the designer or to facilitate rapid convergence to the optimal network. In general, any error criterion may be used that can be formulated quantitatively and the arbitrary inputs then chosen to transform the given network into a more desirable one. The advantage of the continuous transformation over the discrete approach is that the choice of the arbitrary parameters becomes very simple. These techniques have also been applied to active circuits but will be discussed elsewhere.

V. Summary

An extension of the Cauer formulation of the equivalent network theory problem is discussed which applies the transformations directly to the element parameter matrices, thereby placing the elements of the transformed network directly in evidence. The method of steepest descent is applied to choose the arbitrary parameters of the transformation in order to derive optimal networks, that is, networks with minimum number of elements and/or minimum element distribution.

Table I

Summary of Differential Equations for the Resistors

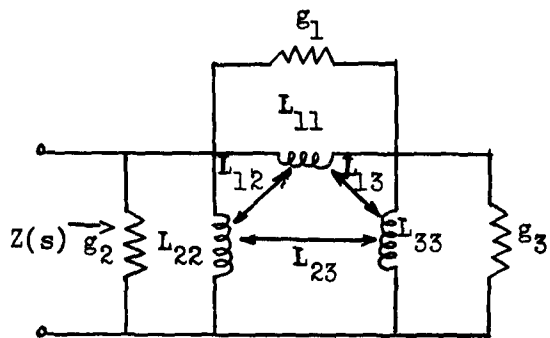| Case	Additional Constraints	Differential Equations
No Element Zero	None	$\dfrac{dg_1}{dx} = g_1(b_2 - b_1) - g_3 b_1$
		$\dfrac{dg_2}{dx} = - g_1(b_2 + b_1) + g_3 b_1$
		$\dfrac{dg_2}{dx} = g_1(b_2 + b_1) + g_3(2b_2 + b_1)$
$g_1 = 0$	$b_1 = 0$	$\dfrac{dg_1}{dx} = 0$
		$\dfrac{dg_2}{dx} = 0$
		$\dfrac{dg_3}{dx} = 2g_3 b_2$
$g_2 = 0$	$b_1(g_3 - g_1) = g_1 b_2$	$\dfrac{dg_1}{dx} = - 2g_1 b_1$
		$\dfrac{dg_2}{dx} = 0$
		$\dfrac{dg_3}{dx} = 2b_1 g_3^2 / g_1$
$g_3 = 0$	$b_2 = - b_1$	$\dfrac{dg_1}{dx} = - 2g_1 b_1$
		$\dfrac{dg_2}{dx} = 0$
		$\dfrac{dg_3}{dx} = 0$

Fig. 1 Example Network
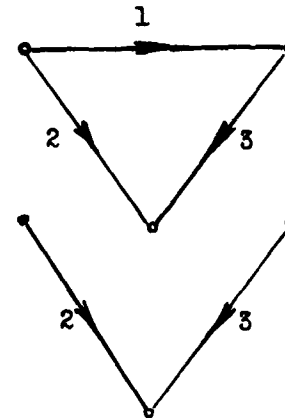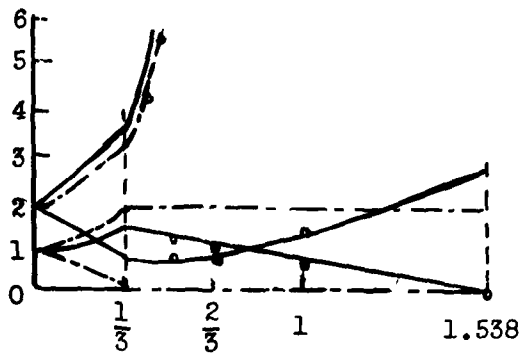

Fig. 2 Graph and Tree
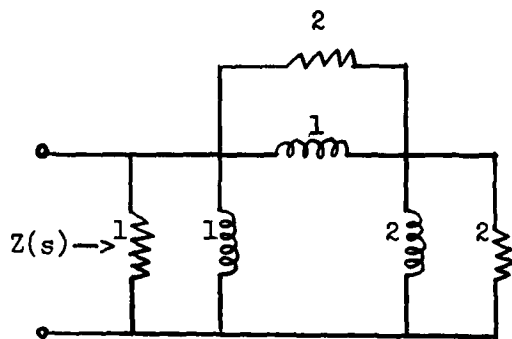
Fig. 3 Numerical Example

$$0 \leq x \leq \frac{1}{3}$$

(a)

Fig. 5 Equivalent Element Versus X

$$g_1 = \frac{2}{3} e^{x-\frac{1}{3}} \qquad L_{11} = 0$$

$$g_2 = \frac{19}{9} - \frac{2}{3} e^{x-\frac{1}{3}} \qquad L_{22} = \frac{5}{3} 2(x-\frac{1}{3})$$

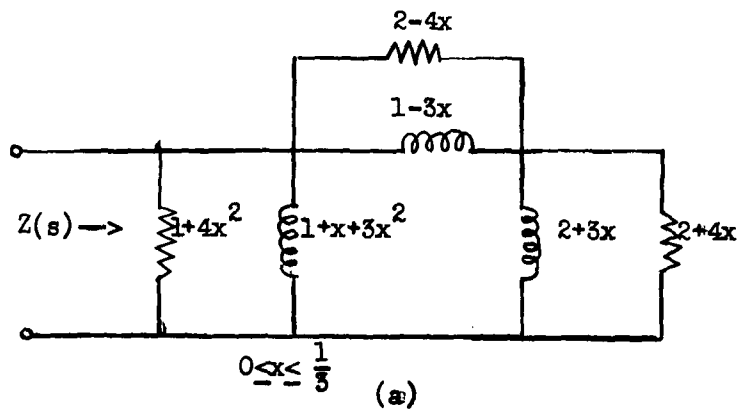$$g_3 = 4 e^{2(x-\frac{1}{3})} - \frac{2}{3} e^{x-\frac{1}{3}} \qquad L_{33} = 3e^{2(x-\frac{1}{3})}$$
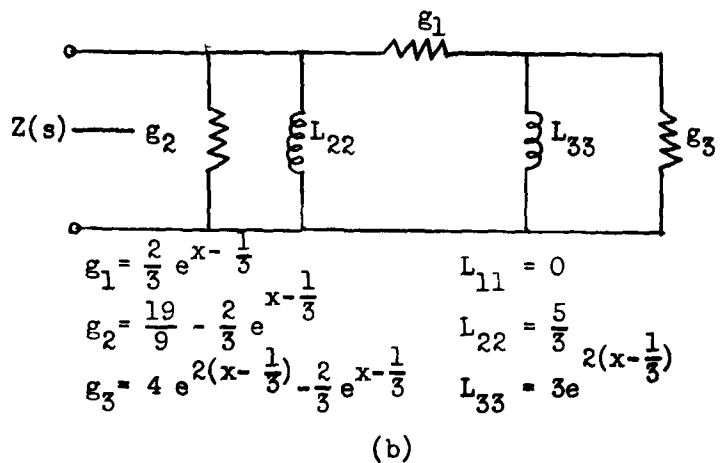
(b)

Fig. 4 Continuous Equivalent Networks

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Urbana, Illinois

Research Laboratory for Electronics
Massachusetts Inst. of Technology
Cambridge 39, Massachusetts

Department of Physics
Columbia University
New York 27, New York

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Microwave Research Laboratory
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Stanford, California

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Electronics Research Lab.
University of California
Berkeley 4, California

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ONR Branch Office
346 Broadway
New York 13, N.Y.

Dr. Nathan Marcuvitz
Polytechnic Institute of
Brooklyn
55 Johnson Street
Brooklyn 1, New York

National Security Agency
Division of Physical Sciences
Ft. George G. Meade,
Maryland

Commanding Officer
ONR Branch Office
495 Summer Street
Boston 10, Massachusetts